The Chain Rule:

Reminder: $y = f(x)$, $x = g(t)$ then y is inderectly a function of t

and $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = f'(x) \cdot g'(t) = f'(g(t)) \cdot g'(t)$ [Case 0]

Case 1: $z = f(x, y)$, $x = g(t)$, $y = h(t)$

then $$\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$$\left(\text{or } \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}\right)$$

Ex: $z = \sqrt{1 + xy}$, $x = \sin t$, $y = t^2$. a) Find $\frac{dz}{dt}$ b) Find $\frac{dz}{dt}$ at $t = \pi$

Sol: (a) $$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} = \underbrace{\frac{y}{2\sqrt{1+xy}}}_{\frac{\partial z}{\partial x}} \underbrace{\cos t}_{\frac{dx}{dt}} + \underbrace{\frac{x}{2\sqrt{1+xy}}}_{\frac{\partial z}{\partial y}} \underbrace{2t}_{\frac{dy}{dt}} = \frac{\frac{t^2}{2}\cos t + t \sin t}{\sqrt{1 + t^2 \sin t}}$$

$\begin{cases} x = \sin t \\ y = t^2 \end{cases}$

(b) $t = \pi \Rightarrow x = 0, y = \pi^2 \rightsquigarrow -\frac{\pi^2}{2}$

Case 2: $z=f(x,y)$ and $x=g(s,t)$, $y=h(s,t)$

then $$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

Ex: $z=(x-y)^5$, $x=s^2t$, $y=st^2$. Find $\frac{\partial z}{\partial s}$, $\frac{\partial z}{\partial t}$.

Sol:

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}=\underbrace{5(x-y)^4}_{\frac{\partial z}{\partial x}}\underbrace{2st}_{\frac{\partial x}{\partial s}}\underbrace{-5(x-y)^4}_{\frac{\partial z}{\partial y}}\underbrace{t^2}_{\frac{\partial y}{\partial s}}=5(x-y)^4(2st-t^2)$$

$$=5(s^2t-st^2)^4(2st-t^2)$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}=\underbrace{5(x-y)^4}_{\frac{\partial z}{\partial x}}\underbrace{s^2}_{\frac{\partial x}{\partial t}}\underbrace{-5(x-y)^4}_{\frac{\partial z}{\partial y}}\underbrace{2st}_{\frac{\partial y}{\partial t}}=5(x-y)^4(s^2-2st)$$

$$\underset{[x=s^2t,\ y=st^2]}{=\!=\!=}5(s^2t-st^2)^4(s^2-2st)$$

General case of Chain Rule:

$$u = u(\underbrace{x_1, \dots, x_n}_{n \text{ variables}}) \quad \text{and for each } j: \quad x_j = x_j(\underbrace{t_1, \dots, t_m}_{m \text{ variables}})$$

then
$$\frac{\partial u}{\partial t_i} = \underbrace{\frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \dots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}}_{n \text{ terms}}$$

Ex: $u = x + yz, \quad x = t, \; y = \ln t, \; z = \sin t \qquad [\, n=3, \; m=1 \,]$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} = 1 \cdot 1 + z\frac{1}{t} + y\cos t$$

$$[z = \sin t,\; y = \ln t] = 1 + \frac{\sin t}{t} + \ln t \cos t$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x + yz) = 1 \qquad \frac{dx}{dt} = \frac{d}{dt}(t) = 1$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x + yz) = z \qquad \frac{dy}{dt} = \frac{d}{dt}(\ln t) = \frac{1}{t}$$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}(x + yz) = y \qquad \frac{dz}{dt} = \frac{d}{dt}(\sin t) = \cos t$$

Ex: Let z be defined implicitly by $x^3+y^3+z^3+6xyz=1$ (*).

(Here we think about z as $z(x,y)$)

Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.

Sol: apply $\frac{\partial}{\partial x}$ to (*), keeping y constant:

$$\frac{\partial}{\partial x}\left(x^3+y^3+(z(x,y))^3+6xy(z(x,y))\right)=\frac{\partial}{\partial x}1$$

$$\underbrace{3x^2}_{\frac{\partial x^3}{\partial x}}+\underbrace{0}_{\frac{\partial y^3}{\partial x}}+\underbrace{3z^2\frac{\partial z}{\partial x}}_{\frac{\partial z^3}{\partial x}}+\underbrace{6yz+6xy\frac{\partial z}{\partial x}}_{\frac{\partial(6xyz)}{\partial x}}=\underbrace{0}_{\frac{\partial 1}{\partial x}},$$

So we have $\quad 3x^2+3z^2\frac{\partial z}{\partial x}+6yz+6xy\frac{\partial z}{\partial x}=0$

$$\Rightarrow \frac{\partial z}{\partial x}=-\frac{3x^2+6yz}{3z^2+6xy}=-\frac{x^2+2yz}{z^2+2xy}$$

apply $\frac{\partial}{\partial y}$ to (*), keeping x constant:

$$3y^2+3z^2\frac{\partial z}{\partial y}+6xz+6xy\frac{\partial z}{\partial y}=0$$

$$\Rightarrow \frac{\partial z}{\partial y}=-\frac{y^2+2xz}{z^2+2xy}$$

Implicit differentiation:

If $y = f(x)$ is defined implicitly, by $\boxed{F(x, \underbrace{y}_{=f(x)}) = 0}$

We can find $\frac{dy}{dx}$ via $0 = \boxed{\frac{d}{dx} 0 = \frac{d}{dx} F(x,y)} = \frac{\partial F}{\partial x} \underbrace{\frac{dx}{dx}}_{=1} + \frac{\partial F}{\partial y} \frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} = -\frac{F_x}{F_y}$$

Ex: y defined by $x^2 + y^3 = 1$. Find $y'(x)$.

Sol: $F(x,y) = x^2 + y^3 - 1 \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x}{3y^2}$

Ex: $z = z(x,y)$ defined by $x^2 + y^5 + z^3 = 0$ $(*)$. Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

Sol: $\frac{\partial}{\partial x}(*)$: $2x + 0 + 3z^2 \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{2x}{3z^2}$

$\frac{\partial}{\partial y}(*)$: $\underbrace{0}_{\frac{\partial x^2}{\partial y}} + \underbrace{5y^4}_{\frac{\partial y^5}{\partial y}} + \underbrace{3z^2 \frac{\partial z}{\partial y}}_{\frac{\partial (z(x,y))^3}{\partial y}} = \underbrace{0}_{\frac{\partial 0}{\partial y}} \Rightarrow \frac{\partial z}{\partial y} = -\frac{5y^4}{3z^2}$