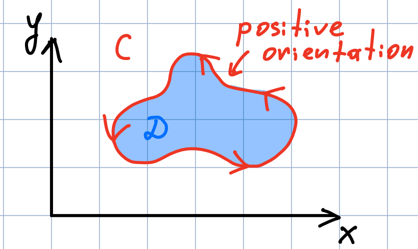
0. Green's thm:

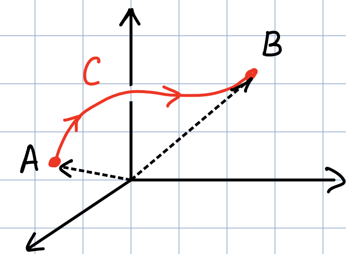
$$\oint_C P\,dx + Q\,dy = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

positively oriented



1. Fund. thm for line integrals:

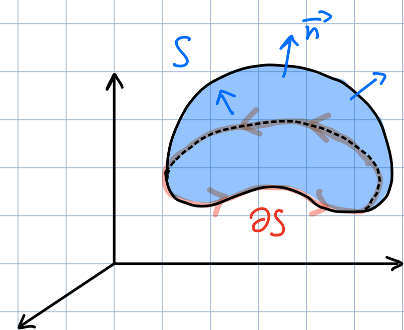
$$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$$



2. Stokes thm:

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{r}$$

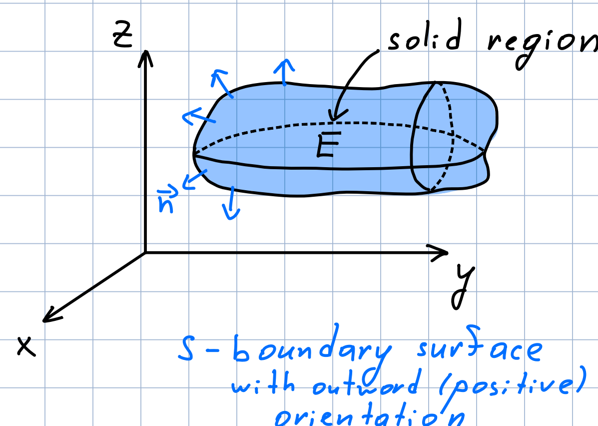
boundary curve



3. Divergence thm:

$$\iiint_E \operatorname{div} \vec{F}\, dV = \iint_{\partial E} \vec{F} \cdot d\vec{S}$$

boundary surface



Question 1. *Let S be the surface defined as $z = 4 - 4x^2 - y^2$ with $z \geq 0$ and oriented upward. Let $\mathbf{F} = \langle x - y, x + y, ze^{xy} \rangle$. Compute* $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$.

Sol: Stokes thm:

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{r}$$

$\uparrow$ boundary curve

1) Need to identify bdry curve:

$z = 0$ i.e. $0 = 4 - 4x^2 - y^2$

$\Rightarrow \quad 4x^2 + y^2 = 4$

$\frac{x^2}{1} + \frac{y^2}{4} = 1$ – ellipse

2) Parametrization

$x = \cos t, \quad y = 2\sin t, \quad 0 \leq t \leq 2\pi$

// $x = a\cos t$, $y = b \sin t$ for $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ //

3) $\oint_{\partial S} (x-y)\,dx + (x+y)\,dy + z e^{xy}\,dz$

$$= \int_0^{2\pi} \Big((\cos t - 2\sin t)(-\sin t) + (\cos t + 2\sin t)\, 2\cos t + 0 \Big) dt$$

$$= \int_0^{2\pi} \underbrace{2\sin^2 t + 2\cos^2 t}_{=2} + 3\underbrace{\sin t \cos t}_{\frac{1}{2}\sin(2t)}\, dt$$

$$= \int_0^{2\pi} \Big(2 + \tfrac{3}{2}\sin(2t) \Big) dt = 2t - \tfrac{3}{4}\cos 2t \Big|_0^{2\pi} = 4\pi$$

Question 2. *Evaluate*

$$\int_C (y^3 + \cos x)dx + (\sin y + z^2)dy + x\,dz$$

where C is the closed curve parametrized by $\mathbf{r}(t) = \langle \cos t, \sin t, \sin 2t\rangle$. (Hint: *the curve C lies on the surface $z = 2xy$.*)

<u>Sol:</u> $\oint_C (y^3 + \cos x)\,dx + (\sin y + z^2)\,dy + x\,dz = \oint_C \langle y^3 + \cos x, \sin y + z^2, x\rangle \cdot d\vec{r}$

1) C is curve in $\mathbb{R}^3 \Rightarrow$ Use Stokes thm: $\oint_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$

(not Green's thm)

boundary curve

2) Use the hint and parametrize S by

$\vec{r}(x,y) = \langle x, y, 2xy\rangle$ with $x, y \in \mathcal{D} = \{(x,y) \mid x^2 + y^2 \le 1\}$

3) Find "positive" normal field

(a) $\vec{r}_x \times \vec{r}_y = \langle -2y, -2x, 1\rangle$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2y \\ 0 & 1 & 2x \end{vmatrix} = \langle -2y, -2x, 1\rangle$$

(b) Note that C is counterclockwise oriented

and $\vec{r}_x \times \vec{r}_y$ points upward $\Rightarrow \vec{n} = \vec{r}_x \times \vec{r}_y$ (Not $\vec{r}_y \times \vec{r}_x$)

4) Find $\operatorname{curl} \vec{F}$:

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^3 + \cos x & \sin y + z^2 & x \end{vmatrix} = \langle -2z, -1, -3y^2\rangle$$

5) Evaluate the integral:

$$\oint_C (y^3 + \cos x)\,dx + (\sin y + z^2)\,dy + x\,dz \overset{\text{stokes}}{=} \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$$

$$= \iint_D \operatorname{curl}\vec{F}(\vec{r}(x,y))\cdot\vec{n}\,dA$$

$$= \iint_D \langle -2\underbrace{z}_{2xy}, -1, -3y^2\rangle\cdot\langle -2y, -2x, 1\rangle\,dA$$

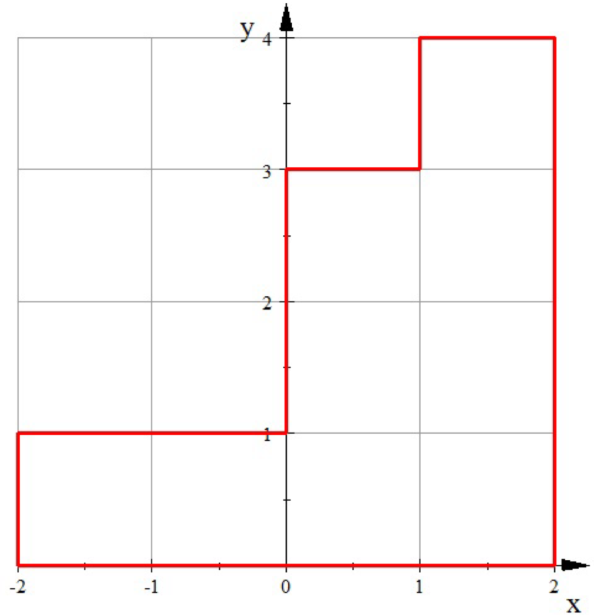
$$= \iint_D (8xy^2 + 2x - 3y^2)\,dA \overset{\text{polar}}{=\!=} \int_0^{2\pi}\underbrace{\int_0^1 (8r^3\cos\theta\sin^2\theta + 2r\cos\theta - 3r^2\sin^2\theta)\,r\,dr}_{\frac{8}{5}r^5\cos\theta\sin^2\theta + \frac{2}{3}r^3\cos\theta - \frac{3}{4}r^4\sin^2\theta\Big|_{r=0}^{r=1}}\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{8}{5}\cos\theta\sin^2\theta + \frac{2}{3}\cos\theta - \frac{3}{4}\underbrace{\sin^2\theta}_{\frac{1-\cos 2\theta}{2}}\right)d\theta$$

$$= \frac{8}{15}\sin^3\theta + \frac{2}{3}\sin\theta - \frac{3\theta}{4\cdot 2} + \frac{3}{16}\sin 2\theta\Big|_0^{2\pi}$$

$$= -\frac{3}{4}\pi$$

Question 3. *Evaluate* $\oint_C (x^4y^5 - 2y)dx + (3x + x^5y^4)dy$ *where* C *is the curve below*



Sol: Green's thm: $\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

1) Find $\frac{\partial Q}{\partial x}$ and $\frac{\partial P}{\partial y}$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(3x + x^5y^4) = 3 + 5x^4y^4 \qquad \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(x^4y^5 - 2y) = 5x^4y^4 - 2$$

2) Evaluate the integral

$$\oint_C (x^4y^5 - 2y)\,dx + (3x + x^5y^4)\,dy = \iint_D 5\,dA = 5 \cdot \text{Area}(D) = 5 \cdot 9$$

// D = union of 9 unit squares //

Question 4. *Calculate the work done by the vector field* $\mathbf{F} = \langle 4y - 3x, x - 4y \rangle$ *on a particle as the particle moves counterclockwise once around the ellipse* $x^2 + 4y^2 = 4$.

Recall work $= \int_C \vec{F} \cdot d\vec{r}$

Sol: 1) C - closed curve in xy-plane => Apply Green's thm

2) $$\oint_C \vec{F} \cdot d\vec{r} = \oint_C \underbrace{(4y-3x)}_{P}dx + \underbrace{(x-4y)}_{Q}dy = \iint_D (\underset{\frac{\partial Q}{\partial x}}{1} - \underset{\frac{\partial P}{\partial y}}{4})\,dA$$

$$= -3\iint_D 1\,dA = -3 \cdot \text{Area}(D)$$

// $\frac{x^2}{2^2} + \frac{y^2}{1^2} = 1$ //

$$= -3(\pi \cdot 2 \cdot 1) = -6\pi$$

Question 5. *Compute* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *where*

$$\mathbf{F} = \langle (2x+z)\cos(x^2+xz), -(z+1)\sin(y+yz), x\cos(x^2+xz) - y\sin(y+yz)\rangle$$

and C *is parametrized by* $\mathbf{r}(t) = \langle t^3, t^2, \pi t - \sin\frac{\pi t}{2}\rangle$, $0 \le t \le 1$.

Sol: 1) We want to use Fundamental thm of line integrals

2) Find a potential function

$$\begin{cases} f_x = (2x+z)\cos(x^2+xz) \\ f_y = -(z+1)\sin(y+yz) \\ f_z = x\cos(x^2+xz) - y\sin(y+yz) \end{cases}$$

$\Longrightarrow f(x,y,z) = \sin(x^2+xz) + g(y,z) \Rightarrow$

$f_y = g_y = -(z+1)\sin(y+yz)$

$\Rightarrow g = \cos(y+yz) + h(z)$

$f_z = x\cos(x^2+xz) - y\sin(y+yz) + h_z$

$\Rightarrow h = \text{const}$

$\Rightarrow f = \sin(x^2+xz) + \cos(y+yz) + \text{const}$

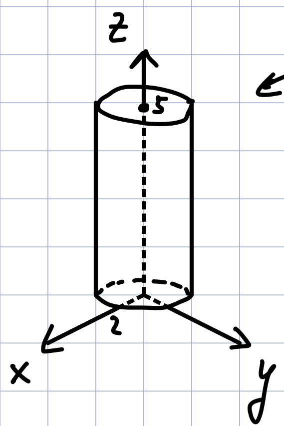
3) Apply F.THM. to evaluate the integral

$$\int_C \nabla f\cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0)) = f(1,1,\pi-1) - f(0,0,0)$$

$$= (\sin\pi + \cos\pi) - (\sin 0 - \cos 0) = -2$$

Question 6. *Compute the flux of* $\mathbf{F} = xyz\mathbf{j}$ *across the cylinder with lateral surface* $x^2 + y^2 = 4$, *top* $z = 5$, *and bottom* $z = 0$, *and outward orientation.*

Sol: $\vec{F} = \langle 0, xyz, 0 \rangle$



← closed surface ⇒ can apply Divergence thm

1) Find $\operatorname{div} \vec{F}$.

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial y} xyz = xz$$

2) Apply Div. thm. to evaluate the integral

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV$$

$$= \iiint_E xz \, dV \underset{\text{cylindrical}}{=\!=}$$

$$\int_0^2 \int_0^5 \underbrace{\int_0^{2\pi} (r\cos\theta) z \, r \, d\theta}_{zr^2 \sin\theta \Big|_0^{2\pi} = 0} \, dz \, dr = 0$$

Question 7. *Compute* $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, *where* $\mathbf{F} = \langle 4xy + z^2, 2x^2 + 6yz, 2xz\rangle$ *and* S *is the closed surface in the first octant bounded by the coordinates planes,* $x = 4$, *and* $z = 9 - y^2$, *given outward orientation.*

Sol: 1) S is closed surface $\Rightarrow$ can apply the Divergence thm

2) Recall: $\operatorname{div}(\operatorname{curl} \vec{F}) = 0$ for any vector-field $\vec{F}$.

3) $$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \iiint_E \underbrace{\operatorname{div}(\operatorname{curl} \vec{F})}_{0} dV = 0$$

Rmk: We also can use Stoke's thm.

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{r}$$

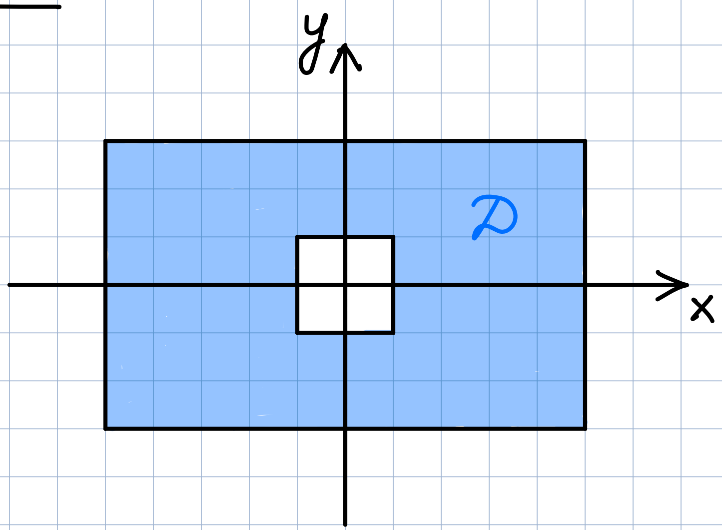
boundary curve

Q: What is the boundary?

S is a closed surface $\Rightarrow \partial S = \phi$ i.e. the line integral is along nothing $\Rightarrow$ it is 0.

Question 8. *Compute* $\oint_C (y-x)dx + (2x-y)dy$ *where* C *is the boundary of the region lying inside the rectangle bounded by* $x=-5$, $x=5$, $y=-3$, *and* $y=3$, *and outside the square bounded by* $x=-1$, $x=1$, $y=-1$, *and* $y=1$.

Sol:



1) Apply Green's thm:

$$\oint_C \overbrace{(y-x)}^{P}dx + \overbrace{(2x-y)}^{Q}dy = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA$$

2) Evaluate the integral:

$$= \iint_{\mathcal{D}} (2-1)\,dA = \iint_{\mathcal{D}} 1\,dA = \text{Area of } \mathcal{D}$$

$$\text{Area}(\mathcal{D}) = 10\cdot 6 - 2\cdot 2 = 56$$

Question 9. *Evaluate* $\int_C (x^2+y^2)dx + 2xy\,dy$ *where* C *is parametrized by* $\mathbf{r}(t) = t^3\mathbf{i} + t^2\mathbf{j}$, $0 \le t \le 2$.

Sol: 1) $\vec{r}(0) = \vec{0}$ $\quad \vec{r}(2) = \langle 8, 4 \rangle$ i.e. C is NOT a closed curve

2) Apply Fund. thm $\rightsquigarrow$ need to find a potential fct.

$$f_x = x^2 + y^2 \implies f = \frac{1}{3}x^3 + xy^2 + g(y)$$

$$f_y = 2xy \qquad f_y = 2xy + g'(y) \implies g(y) = K$$

$$f = \frac{1}{3}x^3 + xy^2 + K$$

3) $$\int_C \nabla f \cdot dr = f(\vec{r}(2)) - f(\vec{r}(0))$$

$$= f(8,4) - f(0,0) = \left(\frac{512}{3} + 128\right) - 0 = \frac{896}{3}$$

Rmk: Can be done directly!

$$\int_C (x^2+y^2)dx + 2xy\,dy = \int_0^2 \left((t^3)^2 + (t^2)^2\right)\underbrace{x'(t)}_{3t^2} + 2t^3t^2\underbrace{y'(t)}_{2t}\,dt$$

$$= \int_0^2 (3t^8 + 7t^6)\,dt = \frac{1}{3}t^9 + t^7\Big|_0^2 = \frac{896}{3}$$

Question 10. *Find the flux of* $\mathbf{F} = x\mathbf{i} + y^2\mathbf{j} - 2yz\mathbf{k}$ *across the surface which is the boundary of the region bounded by* $z = \sqrt{36 - x^2 - y^2}$ *and* $z = 0$, *with outward orientation.*

Sol: Divergence thm:

$$\iiint_E \operatorname{div} \vec{F}\, dV = \iint_{\partial E} \vec{F} \cdot d\vec{S}$$

1) Find $\operatorname{div} \vec{F}$:

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y^2 + \frac{\partial}{\partial z}(-2yz) = 1 + 2y - 2y = 1$$

2) Note the region is a half of the ball centered at O with radius 6.

3) $$\iiint_E \operatorname{div} F\, dV = \iiint_E 1\, dV = \text{Volume}(E) = \frac{1}{2}\left(\frac{4}{3}\pi 6^3\right) = 144\pi$$

Question 11. *Let S be the surface described by $z = e^{1-x^2-y^2}$, $z \geq 1$, oriented with upward normals. Let* $\mathbf{F} = \langle x, y, 2-2z \rangle$ *and compute the flux of* $\mathbf{F}$ *across* S.

Rmks: Not easy to deal with this surface to compute directly. So, we want to apply some thm. Since it is NOT a closed surface, the first guess – Stokes thm:

$$\iint_S \operatorname{curl} \vec{G} \cdot d\vec{S} = \oint_{\partial S} \vec{G} \cdot d\vec{r}$$

⚠ To use Stokes thm we need to find $\vec{G}$ s.t.

$$\operatorname{curl} \vec{G} = \vec{F} \quad (= \langle x, y, 2-2z \rangle)$$

HARD, not always possible

Sol: 1) The boundary of S: $z = 1$, $e^{1-x^2-y^2} = 1 \iff 1 - x^2 - y^2 = 0$

so $z = 1$, $x^2 + y^2 = 1$

2) Consider the disc $\mathcal{D} := \{(x, y, z) : x^2 + y^2 \leq 1, \ z = 1\}$

Note that $S \cup \mathcal{D}$ is a closed surface

3) Let E be solid bounded by $S \cup \mathcal{D}$. – We can apply

Divergence thm!

4) Find $\operatorname{div}\vec{F}$.

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y + \frac{\partial}{\partial z}(2-2z) = 1+1-2=0$$

5) Apply Divergence thm:

$$\iint_{S\cup D} \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV = 0$$

given outward orientation

6) Recall that S is oriented upward $\Rightarrow$ we should orient D downward, then we have:

$$0 = \iint_{S\cup D} \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot d\vec{S} + \iint_D \vec{F}\cdot d\vec{S}$$

$$\Rightarrow \iint_S \vec{F}\, d\vec{S} = -\iint_D \vec{F}\cdot d\vec{S}$$

7) Parametrise D: $\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 1\rangle$

$$0 \le r \le 1, \quad 0 \le \theta \le 2\pi$$

8) $\vec{r}_r = \langle \cos\theta, \sin\theta, 0\rangle$

$\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle 0, 0, r\rangle$$

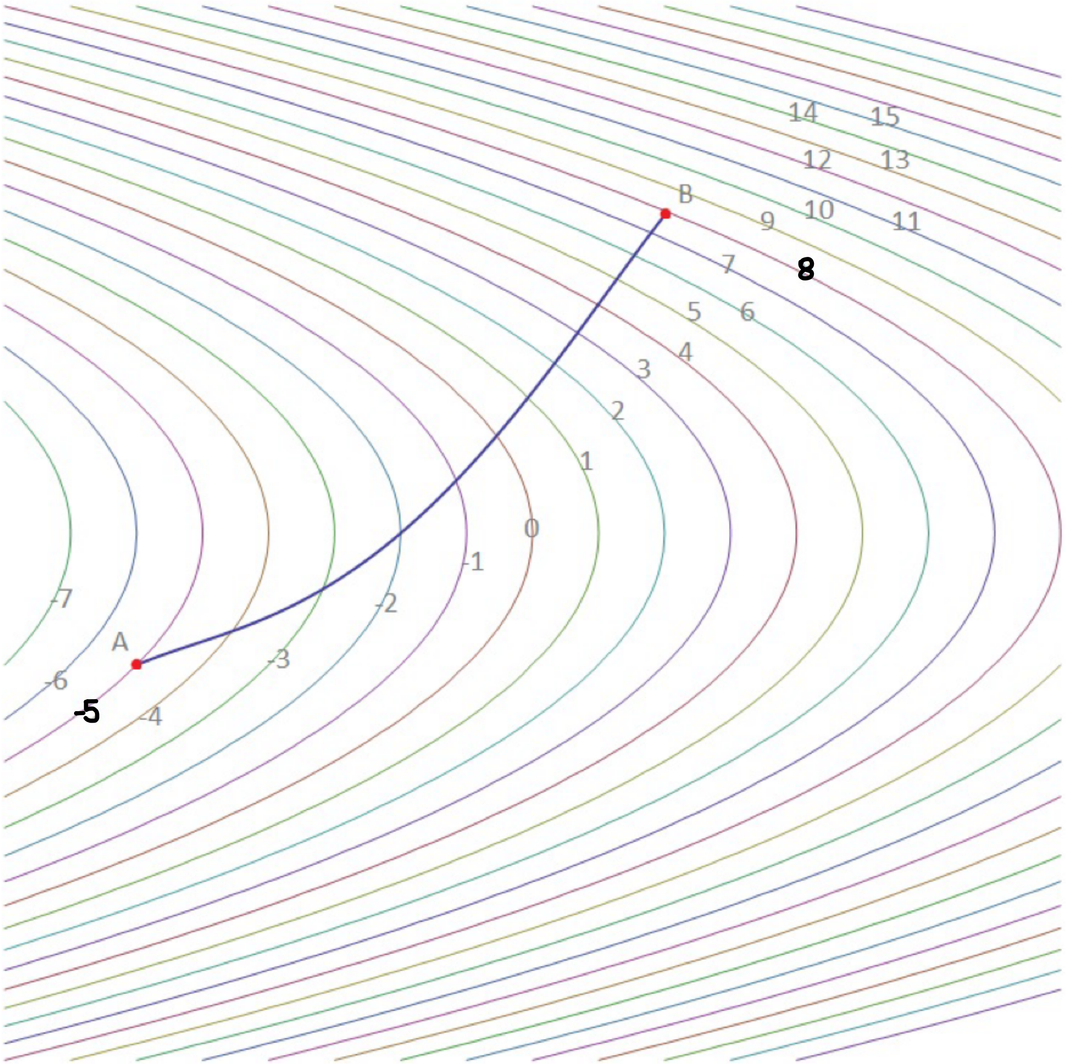
$$\Rightarrow \quad \vec{r}_\theta \times \vec{r}_r = \langle 0, 0, -r \rangle$$ - downward normal vector

9) Evaluate $\iint_{\mathcal{D}} \vec{F} \cdot d\vec{S}$.

$$\iint_{\mathcal{D}} \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^1 \underbrace{\vec{F}(\vec{r}(r,\theta))}_{= \langle r\cos\theta,\, r\sin\theta,\, 2-2\cdot 1\rangle = \langle r\cos\theta,\, r\sin\theta,\, 0\rangle} (\vec{r}_\theta \times \vec{r}_r)\, dr\, d\theta = \int_0^{2\pi}\int_0^1 0\, dr\, d\theta = 0$$

$$\Rightarrow \iint_S \vec{F} \cdot d\vec{S} = 0.$$

Question 12. *Compute* $\int_C \nabla f \cdot d\mathbf{r}$ *where a contour plot of* f *is provided below and* C *is the curve drawn which starts at* A *and ends at* B*.*



Sol: Fund. thm: $\int_C \nabla f \cdot d\vec{r} = f(B) - f(A) = 8 - (-5) = 13$