True or false?

1) Let $\vec{F} = \langle 2x^2+2,\ 3xy,\ ze^x \rangle$, then

(a) $\operatorname{curl} \vec{F} = \langle 4x,\ 3x,\ e^x \rangle$

(b) $\operatorname{div} \vec{F} = \langle 4x,\ 3x,\ e^x \rangle$

(c) $\nabla \vec{F} = \langle 4x,\ 3x,\ e^x \rangle$

Q: Is there a potential function for the vector field $\langle 4x,\ 3x,\ e^x \rangle$?

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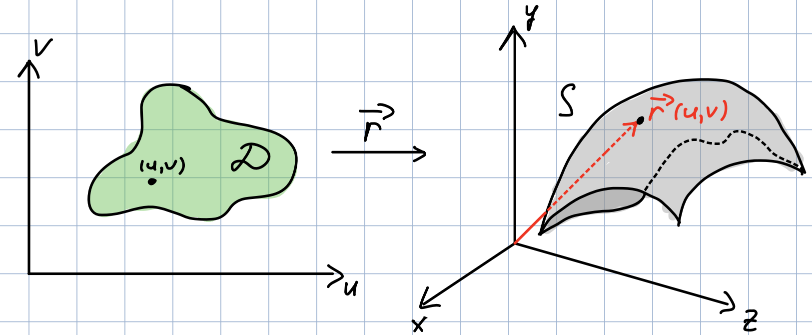
2) Let S be a surface given by $\vec{r}(u,v)$.

(a) $$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \underbrace{dA}_{du\,dv}$$

vector field $\vec{F}(x,y,z)$

in uv-plane



(b) Let S be a graph of some function $g(x,y)$ with upward orient.

Then $$\iint_S \vec{F} \cdot d\vec{S} = \iint_D (-Pg_x - Qg_y + R)\, dA.$$

3) Let $f(x,y,z) = 2x^2 + 3y - e^z$, then

(a) $\iint_S f(x,y,z)\,dS = \iiint_B (4x + 3 - e^z)\,dV$,

where S is a unit sphere surrounding ball B.

(b) One can still apply Divergence thm to evaluate $\iint_S f(x,y,z)\,dS$

Problem:

Let S be a surface given by $\vec{r}(u,v) = \langle uv, u-v, u+v \rangle$ with $1 \le u \le 2$, $2 \le v \le 4$.

Find $\iint_S \vec{F} \cdot d\vec{S}$, where $\vec{F} = \langle 2x+3, y, z^2 + 2xz \rangle$

Solutions

True or false?

1) Let $\vec{F} = \langle 2x^2+2,\ 3xy,\ ze^x \rangle$, then

(a) $\operatorname{curl}\vec{F} = \langle 4x,\ 3x,\ e^x \rangle$

FALSE

$$\operatorname{curl}\vec{F} = "\nabla \times \vec{F}" = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x^2+2 & 3xy & ze^x \end{vmatrix} = \langle 0-0,\ 0-ze^x,\ 3y-0 \rangle$$

$$= \langle 0,\ -ze^x,\ 3y \rangle$$

(b) $\operatorname{div}\vec{F} = \langle 4x,\ 3x,\ e^x \rangle$

FALSE

$$\operatorname{div}F = "\nabla \cdot \vec{F}" = \frac{\partial}{\partial x}(2x^2+2) + \frac{\partial}{\partial y}3xy + \frac{\partial}{\partial z}ze^x = 4x + 3x + e^x$$

(c) $\nabla\vec{F} = \langle 4x,\ 3x,\ e^x \rangle$

FALSE $\quad \nabla\vec{F}$ NOT defined

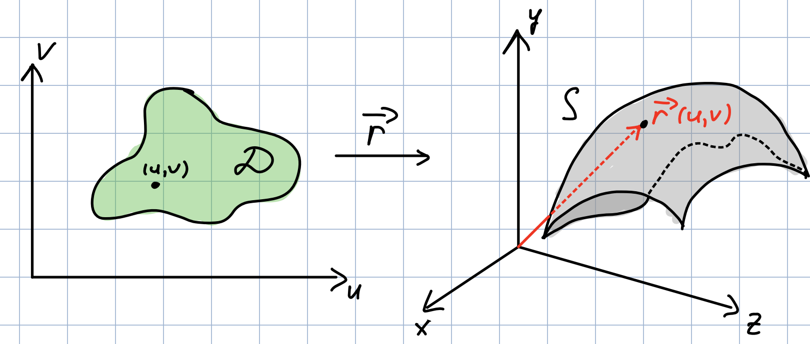
$\nabla f = \langle f_x, f_y, f_z \rangle$ for a FUNKTION $f(x,y,z)$.

Q: Is there a potential function for the vector field $\langle 4x, 3x, e^x \rangle$?

$\frac{\partial}{\partial x}(3x) \neq \frac{\partial}{\partial y}(4x) \Rightarrow$ there is no such a function

$\vec{F} = \langle P, Q, R \rangle$ conservative iff $\operatorname{curl}\vec{F} = 0$ (i.e. $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},\ \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x},\ \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$)

2) Let S be a surface given by $\vec{r}(u,v)$.



(a) $\iint\limits_S \vec{F}\cdot d\vec{S} = \iint\limits_D \vec{F}(\vec{r}(u,v))\cdot \frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}\,\underbrace{dA}_{du\,dv}$

vector field $\vec{F}(x,y,z)$

in uv-plane

FALSE

(*) $\iint\limits_S \vec{F}\cdot d\vec{S} \overset{def}{=\!=} \iint\limits_S \vec{F}\cdot\vec{n}\,dS = \iint\limits_D \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u\times\vec{r}_v)\,dA$

unit normal vertor field for S

Note that $\vec{n} = \frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}$ and $dS = |\vec{r}_u\times\vec{r}_v|\,dA$

Q: Does the orientation of S in (*) have to be upward?

No, it should be given by $\vec{r}_u\times\vec{r}_v$.

(b) Let S be a graph of some function $g(x,y)$ with upward orient.

Then $\iint\limits_S \vec{F}\cdot d\vec{S} = \iint\limits_D (-Pg_x - Qg_y + R)\,dA$.

TRUE

Recall, that for $\vec{r}(x,y) = \langle x, y, g(x,y)\rangle$ $\vec{r}_x\times\vec{r}_y = \langle -g_x, -g_y, 1\rangle$

3) Let $f(x,y,z) = 2x^2 + 3y - e^z$, then

(a) $\iint_S f(x,y,z)\,dS = \iiint_B (4x + 3 - e^z)\,dV$,

where S is a unit sphere surrounding ball B.

FALSE The Divergence thm: $\iint_S \vec{F} \cdot d\vec{S} = \iiint_B \operatorname{div} \vec{F}\,dV$

(b) One can still apply Divergence thm to evaluate $\iint_S f(x,y,z)\,dS$

TRUE

Recall that $\iint_S \vec{F} \cdot d\vec{S} \overset{def}{=\!=} \iint_S \vec{F} \cdot \vec{n}\,dS$

vector field $\vec{F}(x,y,z)$

unit normal vector field for S

The unit normal vector to unit sphere S: $\vec{n} = \langle x, y, z \rangle$

WANTED: $\vec{F} \cdot \vec{n} = f(x,y,z) = 2x^2 + 3y - e^z$

$$Px + Qy + Rz = 2x^2 + 3y - e^z$$

Consider $\vec{F} = \langle 2x, 3, -\frac{e^z}{z} \rangle$. Now we can apply Divergence thm

$$\iint_S f(x,y,z)\,dS = \iint_S \vec{F} \cdot \vec{n}\,dS = \iint_S \vec{F} \cdot d\vec{S} = \iiint_B \operatorname{div} \vec{F}\,dV$$

Ex: Let S be a surface given by $\vec{r}(u,v) = \langle uv, u-v, u+v \rangle$ with $1 \le u \le 2$, $2 \le v \le 4$.

Find $\iint_S \vec{F} \cdot d\vec{S}$, where $\vec{F} = \langle 2x+3, y, z^2 + 2xz \rangle$

Sol: $\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, dA$, $D: \begin{cases} 1 \le u \le 2 \\ 2 \le v \le 4 \end{cases}$

$$\vec{F}(\vec{r}(u,v)) = \langle 2uv+3, u-v, (u+v)^2 + 2uv(u+v) \rangle$$

$$\vec{r}_u = \langle v, 1, 1 \rangle$$
$$\vec{r}_v = \langle u, -1, 1 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle 1-(-1), u-v, -v-u \rangle = \langle 2, u-v, -u-v \rangle$$

$$\iint_D 4uv + 6 + (u-v)^2 - (u+v)((u+v)^2 + 2uv(u+v))\, dA$$

$$= \int_1^2 \int_2^4 4uv + 6 + u^2 - 2uv + v^2 - (u+v)(u^2 + 2uv + v^2 + 2u^2v + 2uv^2)\, dv\, du$$

$$= \ldots$$