Review

Vector valued functions and space curves

Vector valued functions

$$\boxed{\vec{r}(t) = \langle f(t), g(t), h(t) \rangle}$$

Vector (-valued) function: domain $\longrightarrow$ range

domain: set of real numbers; range: set of vectors

$$t \longmapsto \vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k} = \langle f(t), g(t), h(t) \rangle$$

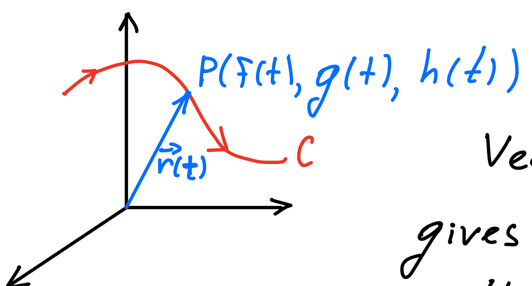
Rmk: domain = values t for which $\vec{r}(t)$ is defined.

Space curves

Set of points (x, y, z) with $x = f(t)$, $y = g(t)$, $z = h(t)$, $t \in I$ (interval) – "space curve" C.

(given functions: f, g, h)

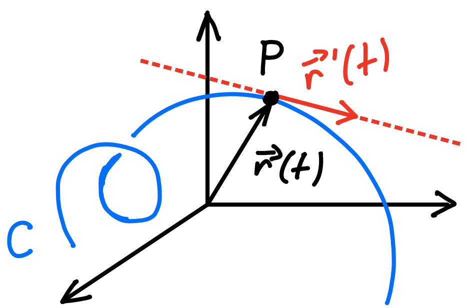
$x = f(t)$, $y = g(t)$, $z = h(t)$: parametric equations of C, t – parameter



Vector function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ gives a "moving" vector, whose tip traces the curve C.

Calculus of space curves

Derivatives: $\vec{r}(t)$ vector function $\rightsquigarrow$ $\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$



- $\vec{r}\,'(t)$ – tangent vector to curve C at point P (if $\vec{r}\,'(t) \neq 0$ and exists)
- tangent line to C at P – line through P, parallel to $\vec{r}\,'(t)$

Integrals:

$$\int_a^b \vec{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\vec{i} + \left(\int_a^b g(t)\,dt\right)\vec{j} + \left(\int_a^b h(t)\,dt\right)\vec{k}$$

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$$\boxed{\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle}$$

Fund. Thm of Calculus (in vector setting)

$$\int_a^b \vec{r}(t)\,dt = \vec{R}(t)\Big|_a^b = \vec{R}(b) - \vec{R}(a)$$

$\vec{R}\,'(t) = \vec{r}(t)$

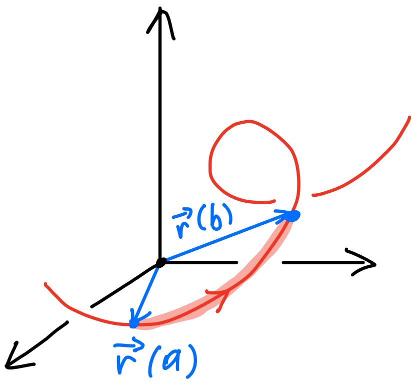
We denote $\vec{R}(t) = \int \vec{r}(t)\,dt$ (indefinite integral)

Arc length & TNB Frame

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, dt$$

length of the curve

$$= \int_a^b |\vec{r}\,'(t)|\, dt$$



$C : \vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

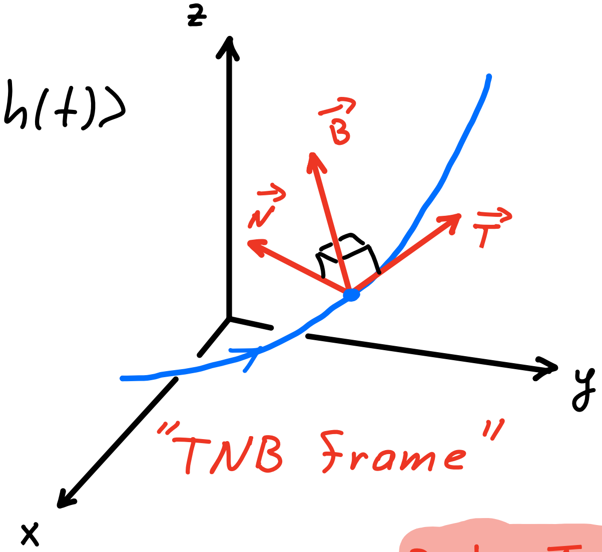
unit

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}$$

normal

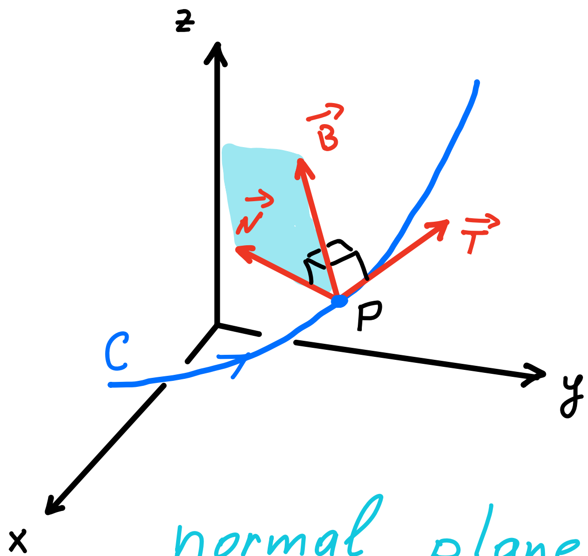
$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

binormal



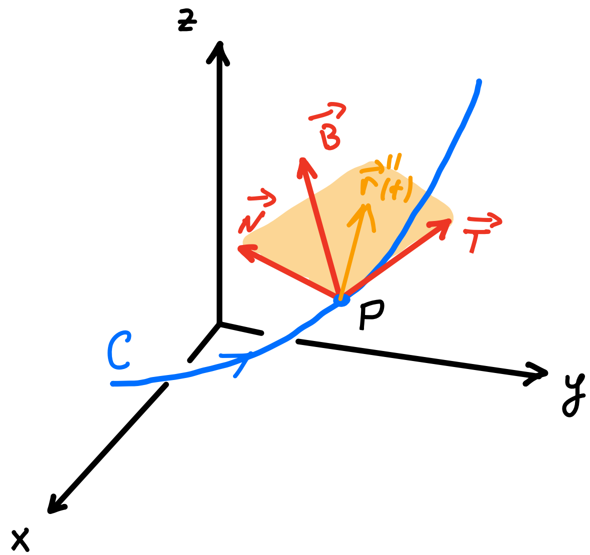
"TNB frame"

Rmk: $T \times N = B$
$T = N \times B$
" $\times T = N = B \times$ "
$N = B \times T$



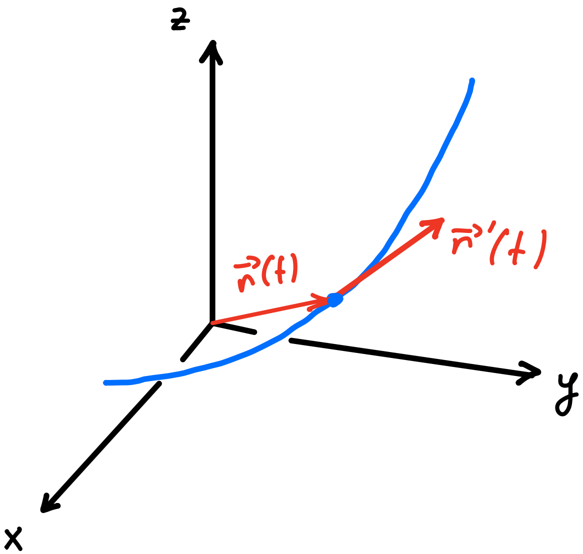
normal plane

osculating plane



Rmk: $\vec{r}\,''(t)$ is in the osculating plane.

Motion in space



particle moving through space;
position at time t: $\vec{r}(t)$

$$\vec{r}\,'(t) = \lim_{h\to 0} \underbrace{\frac{\vec{r}(t+h)-\vec{r}(t)}{h}}_{\text{displacement vector per unit time}} = \vec{v}(t)$$

$\uparrow$ velocity vector

speed: $|\vec{v}(t)| = |\vec{r}\,'(t)| = \frac{ds}{dt}$ ← arc length fct.

↑ rate of change of dist. w.r.t. time

<u>Reminder:</u> $s(t) = \int_a^t |r'(u)|\,du \implies \frac{ds}{dt} = |r'(t)|$

acceleration: $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$

<u>Rmk:</u> If we know $\vec{a}(t)$ and $\vec{r}(t_0)$, $\vec{v}(t_0)$ then
(initial) position and velocity)

$$\vec{v}(t) = \int_{t_0}^{t} \vec{a}(u)\,du + \vec{v}(t_0), \qquad \vec{r}(t) = \int_{t_0}^{t} \vec{v}(u)\,du + \vec{r}(t_0)$$

True-False Quiz

True-False Quiz: 1. Let $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle.$$

2. $\vec{r}(t) = t^3\vec{i} + 2t^3\vec{j} + 3t^3\vec{k}$ is a line.

3. The curve $\vec{r}(t) = \langle 2t, 3-t, 0 \rangle$ is a line passes through the origin.

4. $(\vec{u}(t) \times \vec{v}(t))' = \vec{u}'(t) \times \vec{v}'(t)$

5. The binormal vector $\vec{B}(t) = \vec{N}(t) \times \vec{T}(t)$

6. If $|\vec{r}(t)| = 1$ for all t, then $\vec{r}'(t)$ is orthogonal to $\vec{r}(t)$ for all t

7. If $|\vec{r}(t)| = 1$ for all t, then $|\vec{r}'(t)|$ is a constant

8. $$\vec{r}''(t) = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{|\vec{r}'(t)|}\vec{T}(t) + \frac{\vec{r}'(t) \times \vec{r}''(t)}{|\vec{r}'(t)|}\vec{N}(t)$$

Quiz Answers

True-False Quiz:

Let $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

1. $\int_a^b \vec{r}(t)\,dt = \langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \rangle$. $\boxed{\text{TRUE}}$

2. $\vec{r}(t) = t^3\vec{i} + 2t^3\vec{j} + 3t^3\vec{k}$ is a line. $\boxed{\text{TRUE}}$

This is a line through point $P(0,0,0)$, parallel to the vector $\vec{v} = \langle 1, 2, 3 \rangle$. To see this denote $s = t^3$, then

$x = 0 + 1\cdot s$; $y = 0 + 2\cdot s$; $z = 0 + 3\cdot s$

3. The curve $\vec{r}(t) = \langle 2t, 3-t, 0 \rangle$ is a line passes through the origin. $\boxed{\text{FALSE}}$

It is a line, but the origin $(0,0,0)$ does NOT lie on this line.

4. $(\vec{u}(t) \times \vec{v}(t))' = \vec{u}'(t) \times \vec{v}'(t)$ $\boxed{\text{FALSE}}$

$(\vec{u}(t) \times \vec{v}(t))' = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$ // product rule //

5. The binormal vector $\vec{B}(t) = \vec{N}(t) \times \vec{T}(t)$ $\boxed{\text{FALSE}}$

$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ [which is $= -\vec{N}(t) \times \vec{T}(t)$]

6. If $|\vec{r}(t)|=1$ for all t, then $\vec{r}'(t)$ is orthogonal to $\vec{r}(t)$ for all t. $\boxed{\text{TRUE}}$

$\vec{r}(t)\cdot\vec{r}(t)=|\vec{r}(t)|=1 \Rightarrow \frac{d}{dt}(\vec{r}(t)\cdot\vec{r}(t))=\frac{d}{dt}(1)=0$

Also $\frac{d}{dt}(\vec{r}(t)\cdot\vec{r}(t))=\vec{r}'(t)\cdot\vec{r}(t)+\vec{r}(t)\cdot\vec{r}'(t)=2\vec{r}(t)\cdot\vec{r}'(t)$

$\Rightarrow 2\vec{r}(t)\cdot\vec{r}'(t)=0 \Rightarrow \vec{r}(t)\cdot\vec{r}'(t)=0 \Rightarrow \vec{r}(t)\perp\vec{r}'(t)\ \forall t$

7. If $|\vec{r}(t)|=1$ for all t, then $|\vec{r}'(t)|$ is a constant $\boxed{\text{FALSE}}$

We can stay on unit sphere BUT move with different speed.

// $\vec{r}(t)=\langle\cos(t^2),\sin(t^2),0\rangle$ //

8. Let $\vec{r}(t)$ be a position of a particle at time t. Then the acceleration $\vec{a}(t)$ satisfies

this is a vector! we should take its magnitude

$$\vec{a}(t)=\frac{\vec{r}'(t)\cdot\vec{r}''(t)}{|\vec{r}'(t)|}\vec{T}(t)+\frac{\overbrace{\vec{r}'(t)\times\vec{r}''(t)}}{|\vec{r}'(t)|}\vec{N}(t)$$ $\boxed{\text{FALSE}}$

$$\vec{a}=\underbrace{a_T}_{\text{tangental component}}\vec{T}+\underbrace{a_N}_{\text{normal component}}\vec{N}$$

$$a_T=\vec{a}\cdot\vec{T}=\frac{\vec{r}'(t)\cdot\vec{r}''(t)}{|\vec{r}'(t)|}$$

$$a_N=\vec{a}\cdot\vec{N}=\frac{|\vec{r}'(t)\times\vec{r}''(t)|}{|\vec{r}'(t)|}$$

Few more problems

Problems:

1. Describe by a vector equation the intersection C of $z = x^2 + y^2$ - paraboloid and $y = x^2$ - parabolic cylinder

2. C: $x = 2\cos t$, $y = \sin t$, $z = t$. Find eq. of the tangent line to C at point $(0, 1, \frac{\pi}{2})$.

3. $\vec{r}(t) = \langle t^2, \frac{2}{3}t^3, t \rangle$. Find $\vec{B}$ at $P(1, \frac{2}{3}, 1)$.

4. Find equations of normal and osculating planes for $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\vec{k}$ at $P(0, 1, \frac{\pi}{2})$

Solutions

Ex: Describe by vec. eq. intersection C of $z=x^2+y^2$ - paraboloid

and $y=x^2$ - parabolic cylinder

Sol: Set $t=x$ then $y=t^2$ and $z=x^2+y^2=t^2+t^4$

$\Rightarrow \vec{r}(t)=\langle t, t^2, t^2+t^4\rangle$

Q: Why we started with $x=t$?

In this problem, z depends on x and y

$(z=x^2+y^2)$,

and y depends on x,

so it is natural to start with $x=t$.

Now $\begin{cases} y=x^2 \\ x=t \end{cases} \Rightarrow y=t^2$, then $z=x^2+y^2 \Rightarrow$

$$z=\underbrace{t}_{x}{}^2+(\underbrace{t^2}_{y})^2.$$

Ex: $C: x = 2\cos t,\ y = \sin t,\ z = t$. Find eq. of the tangent line to C at point $(0, 1, \frac{\pi}{2})$

corresponds to $t = \frac{\pi}{2}$

Sol: $\vec{r}(t) = \langle 2\cos t, \sin t, t \rangle$

$\vec{r}\,'(t) = \langle -2\sin t, \cos t, 1 \rangle \Rightarrow \vec{r}\,'(\frac{\pi}{2}) = \langle -2, 0, 1 \rangle$

The tangent line passes through $(0, 1, \frac{\pi}{2})$ and parallel to $\langle -2, 0, 1 \rangle$

$x = 0 - 2t = -2t$

$y = 1 + 0t = 1$

$z = \frac{\pi}{2} + t$

Ex: $\vec{r}(t) = \langle t^2, \frac{2}{3}t^3, t \rangle$. Find $\vec{B}$ at $P(1, \frac{2}{3}, 1)$.

Sol: $P(1, \frac{2}{3}, 1)$ corresp. to $t = 1$

$$\vec{r}\,'(t) = \langle 2t, 2t^2, 1 \rangle \qquad \vec{r}\,''(t) = \langle 2, 4t, 0 \rangle$$

$$\vec{r}\,'(1) = \langle 2, 2, 1 \rangle \qquad \vec{r}\,''(1) = \langle 2, 4, 0 \rangle$$

$\vec{b} = \vec{r}\,'(1) \times \vec{r}\,''(1)$ – normal to osculating plane

$$\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 1 \\ 2 & 4 & 0 \end{vmatrix} = -4\vec{i} + 2\vec{j} + (8-4)\vec{k} = \langle -4, 2, 4 \rangle$$

$$\vec{B}(1) = \frac{\vec{b}}{|\vec{b}|} = \frac{\langle -4, 2, 4 \rangle}{\sqrt{(-4)^2 + 2^2 + 4^2}} = \langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \rangle$$

Ex: Find equations of normal and osculating planes for $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\,\vec{k}$ at $P(0, 1, \frac{\pi}{2})$

Sol: 1) $P(0, 1, \frac{\pi}{2})$ corresponds to $t = \frac{\pi}{2}$ // $\langle \cos(\frac{\pi}{2}), \sin(\frac{\pi}{2}), \frac{\pi}{2} \rangle = \langle 0, 1, \frac{\pi}{2} \rangle$

$$\left.\begin{array}{l} \vec{r}\,'(t) = (\cos t)'\,\vec{i} + (\sin t)'\,\vec{j} + (t')\,\vec{k} = -\sin t\,\vec{i} + \cos t\,\vec{j} + \vec{k} \\ |\vec{r}\,'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2} \end{array}\right\} \Rightarrow \vec{T}(t) = \frac{-\sin t\,\vec{i} + \cos t\,\vec{j} + \vec{k}}{\sqrt{2}}$$

2) normal plane – passes through P, has normal vector $\vec{T}(\frac{\pi}{2})$ (or just $\vec{r}\,'(\frac{\pi}{2})$)

$$\left.\begin{array}{l} P(0, 1, \frac{\pi}{2}) \\ \vec{n} = \vec{r}\,'(\frac{\pi}{2}) = \langle -1, 0, 1 \rangle \end{array}\right\} \quad -1(x-0) + 0(y-1) + 1(z - \tfrac{\pi}{2}) = 0$$

or $-x + z - \frac{\pi}{2} = 0$

3) osculating plane – passes through P, has normal vector $\vec{B}(\frac{\pi}{2})$

4) $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$, we need to find $\vec{N}(t) = \dfrac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}$

$\vec{T}\,'(t) = \frac{1}{\sqrt{2}}\left(-\cos t\,\vec{i} - \sin t\,\vec{j} + 0\vec{k}\right) = -\dfrac{\cos t\,\vec{i} + \sin t\,\vec{j}}{\sqrt{2}}$; $|\vec{T}\,'(t)| = \frac{1}{\sqrt{2}} \Rightarrow \vec{N}(t) = -\cos t\,\vec{i} - \sin t\,\vec{j}$

5) $$\vec{B}(t) = \frac{1}{\sqrt{2}} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{1}{\sqrt{2}}\left(\sin t\,\vec{i} - \cos t\,\vec{j} + \vec{k}\right)$$

6) $P(0, 1, \frac{\pi}{2})$ and $\vec{n} = \sqrt{2}\,\vec{B}(\frac{\pi}{2}) = \langle 1, 0, 1 \rangle$

$$1(x-0) + 0(y-1) + 1(z - \tfrac{\pi}{2}) = 0$$

or $x + z - \frac{\pi}{2} = 0$